Phase transitions in the Potts spin glass model

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We have studied the Potts spin glass with 2-state Ising spins and s-state Potts variables using a cluster Monte Carlo dynamics. The model recovers the $\pm J$ Ising spin glass (SG) for $s = 1$ and exhibits for all s a SG transition at $T_{SG}(s)$ and a percolation transition at higher temperature $T_p(s)$. We have shown that for all values of $s \neq 1$ at $T_p(s)$ there is a thermodynamical transition in the universality class of a ferromagnetic s-state Potts model. The efficiency of the cluster dynamics is compared with that of standard spin flip dynamics.

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I. INTRODUCTION

In nature there are many examples of glassy systems, *i.e.* complex systems that exhibit a very slow dynamics that prevents them to reach the equilibrium state. In this class there is a large variety of systems, like real glasses, spin glasses, supercooled liquids, polymers, granular material, colloids, ionic conductors, orientational glasses, vortex glasses [1]. A common feature of these systems is the *frustration*, that is a competition due to geometry or energy constrains.

Experiments on glasses [2] show that in a temperature driven transition precursor phenomena starts at a temperature well above the ideal glass transition. This temperature depends on experimental conditions and is the onset of some dynamical anomalies, like non-exponential relaxation functions, anomalous diffusion, cooling rate dependent density. Analogous phenomena are present in spin glasses (SG) [3], that are a generalization of Ising model where random distributed ferromagnetic and antiferromagnetic interactions give rise to frustration. In particular, experiments [4] and numerical simulations [5] have shown non-exponential autocorrelation functions below a temperature T^* well above the transition temperature T_{SG}, while above T^* only exponential relaxation functions are seen.

To explain this phenomenon, Randeria *et al.* [6] have suggested that in the SG the non-exponential regime starts at the Griffiths temperature T_c, *i.e.* the critical temperature of the ferromagnetic model, due to the presence of randomly large unfrustrated regions, allowed by the disorder in the SG interactions. Campbell *et al.* [7] have proposed an alternative hypothesis in which the onset T^* of non-exponential behavior coincides with the percolation temperature T_p of Fortuin-Kasteleyn [8]

Coniglio-Klein [9] (FK-CK) clusters (defined below). The idea of Ref. [7] is that in the SG the accessible phase space is simply connected above T_p, while it is not below T_p. Therefore for $T < T_p$ a local Monte Carlo (MC) dynamics performs a random walk on a ramified percolating-like structure with many time scales, giving rise to non-exponential relaxation functions.

Since T_p is less than, but close to, T_c and T^* is difficult to localize, it is not possible to exclude the $T^* = T_c$ hypothesis. However numerical results [10,11] on fully frustrated models, without disorder, where the Randeria *et al.* argument does not apply, are consistent with the $T^* = T_p$ scenario. In Refs. [10,11] this result is given both for spin-flip and bond-flip dynamics. The latter is strictly related to the bond *frustrated percolation* (FP) where *frustrated loops*, *i.e.* closed paths of connected bonds covering an odd number of negative interactions, have zero weight [10]. The FP model and the $\pm J$ SG model can be recovered as particular cases of a $2s$-state Potts spin glass (PSG) [12] respectively for $s = 1/2$ and $s = 1$.

In this paper we will show numerical results on the static properties of the PSG model in 2D. For any s the model exhibits a SG transition at a temperature $T_{SG}(s)$ (in 2D $T_{SG} = 0$ for any s) and a FK-CK percolation transition at higher temperature $T_p(s)$. For $s \neq 1$ the higher transition corresponds to a real thermodynamical transition of a s-state Potts model [13]. To make the algorithm faster we use the Swendsen and Wang (SW) [14] cluster MC dynamics that prevent the slowing down for temperatures near $T_p(s)$. Dynamical properties of the model for $s = 2$ are given in Ref. [15], where we have shown that autocorrelation times diverge at T_p and, as in SG model, non-exponential relaxation functions are present below T_p.

These results on SG model [7], FP model [10], fully frustrated systems [11] and PSG model [15] suggest that the FK-CK percolation may play a role in the context of precursor phenomena, since below T_p the frustration is present at all length scales by means of FK-CK percolating cluster, that cannot include frustrated loops.

In Sec. II we define the Hamiltonian and review some theoretical results. In Sec. III we present the numerical results for the model in finite dimensions. In Sec. IV we define the SW dynamics and compere it with the spin flip dynamics, verifying the efficiency of the cluster dynamics. In Sec. V we give the conclusions.

II. HAMILTONIAN FORMALISM

The Potts spin glass (PSG) model is defined by the Hamiltonian

$$H = -sJ \sum_{\langle i,j \rangle} [\delta_{\sigma_i \sigma_j}(\epsilon_{i,j} S_i S_j + 1) - 2] \tag{1}$$

where to each lattice site is associated an Ising spin $S_i = \pm 1$ and a s-state Potts spin $\sigma_i = 1, \ldots, s$. The sum is extended over all NN sites, $\epsilon_{i,j} = \pm 1$ is a random quenched variable and J is the strength of interaction. The model is a superposition of a ferromagnetic s-state Potts model [13] and a $\pm J$ Ising SG [3] and for $\delta_{\sigma_i \sigma_j} = 1$ (i.e. $s = 1$) recovers the $\pm J$ Ising SG Hamiltonian.

Following Ref. [12] it is possible to define FK-CK clusters on this model, activating a bond, between NN sites with both SG interaction and Potts interaction satisfied, with probability

$$p = 1 - e^{-2sJ/k_B T} , \tag{2}$$

and defining a cluster as the maximal set of connected bonds.

For a given set of interaction $\{\epsilon_{i,j}\}$ it is possible to shown [12] that Z can be expressed in terms of bond configurations C

$$Z\{\epsilon_{i,j}\} = \sum_{\{S_i, \sigma_i\}} e^{-H/k_B T} = \sum_C W_s(C) \tag{3}$$

where $W_s(C) = 0$ if C includes any frustrated loop, otherwise

$$W_s(C) = p^{|C|}(1-p)^{|A|}(2s)^{N(C)} \tag{4}$$

where p is given in eq. (2), $N(C)$ is the number of clusters in the configuration C, $|C|$ is the number of bonds and $|C| + |A|$ is total number of interactions. Let us observe that, while the Hamiltonian (1) is defined only for integer values of s, the eq. (3) is meaningful for every values of s and that for $s = 1/2$ the eq. (3) gives the partition function of bond FP where a bond configuration without frustrated loops has a weight $W(C) = e^{\beta \mu |C|}$ with $\beta \mu = \ln(e^{\beta J} - 1)$, while a bond configuration with frustrated loops has a zero weight. Furthermore in the limit $s \to 0$ eq. (3) gives the partition function of the *tree percolation* [13] where any bond configuration with a loop is excluded.

Exact renormalization group (RG) analysis on hierarchical lattice [16] has predicted for the PSG two critical temperatures. The lower temperature $T_{SG}(s)$ corresponds to a SG transition in the universality class of $\pm J$ Ising SG, and the higher $T_p(s)$ to a percolation transition in the universality class of a ferromagnetic s-state Potts model. Same results are given for the fully frustrated version of the model studied with a mean field approach [17].

Looking at the partition function (3) one should expect a singularity at $T_p(s)$, for any $s \neq 0$, due to the singularity in the number of cluster $N(C)$. Nevertheless this singularity has never been observed in the case of SG ($s = 1$). In fact, the RG calculations in the case $s = 1$ show a singularity at T_{SG} for the SG free energy and no singularity at $T_p(1)$. This result is interpreted in Ref. [16] supposing that the free energy of the Hamiltonian (1) has the form

$$F_s(T) - F_s(T_p) \sim A(s)(T - T_p(s))^{2-\alpha(s)} \tag{5}$$

where $A(s)$ is an amplitude which vanishes for $s \to 1$ and $\alpha(s)$ is the specific heat exponent.

III. MONTE CARLO RESULTS

We have done our simulation using the Swendsen and Wang (SW) Monte Carlo cluster dynamics [14] described in Sec. IV. As we will show, the SW dynamics is faster then standard local dynamics, like spin flip dynamics, but suffers of slowing down near the SG critical temperature. Nevertheless, since the SG transition in 2D occurs at $T_{SG} = 0$ [18] and we are interested in study the system near the percolation transition at $T_p > T_{SG}$, the SW dynamics is particularly indicated.

We have performed numerical simulation of the PSG model for $s = 2, 7, 50$ on a 2D square lattice with linear sizes ranging from $L = 10$ to 60 lattice steps and with quenched random interaction configurations $\{\epsilon_{ij}\}$. Defining as MC step an update of all the spins of the system, we have discarded the data of the first 7 500 MC steps and have collected data over 15 000, 25 000 or 50 000 MC steps, depending on the temperatures and sizes.

For each s we have calculated the Binder parameter for the energy density E [19] defined as

$$V = 1 - \frac{\langle E^4 \rangle}{3\langle E^2 \rangle^2} \tag{6}$$

where the symbol $\langle \cdot \rangle$ stands for the thermal average. This quantity allows to localize the transition and to distinguish between first order and second order phase transition. In fact, for a second order phase transition in the limit $L \to \infty$ it is $V = 2/3$ for all temperatures, while for a first order phase transition it is

$$\frac{2}{3} - V^{\min} = \frac{1}{3} \frac{(E_+ - E_-)^2(E_+ + E_-)^2}{(E_+^2 + E_-^2)^2} \tag{7}$$

where $V^{\min}$ is the minimum value of V (occurring at the phase transition temperature) and $E_+ - E_-$ is the energy jump, related to the latent heat, at the same temperature.

To estimate the thermodynamical critical exponents for the second order phase transition we have measured the Potts order parameter

$$M = \frac{s \, \max_i(M_i) - 1}{s - 1} \qquad (8)$$

(where $i = 1, \ldots s$, M_i is the density of Potts spins in the ith state), the susceptibility

$$\chi = \left[\frac{\langle M^2 \rangle - \langle M \rangle^2}{N} \right]^{1/2} \qquad (9)$$

(where N is the total number of Potts spins) and the specific heat

$$C_H = \left[\frac{\langle E^2 \rangle - \langle E \rangle^2}{N} \right]^{1/2} . \qquad (10)$$

Furthermore to estimate the percolation critical exponents we have calculated the percolation probability per spin

$$P = 1 - \sum_k k n_k \ , \qquad (11)$$

where k is the cluster size and n_k is the density of clusters of size k, the mean cluster size

$$S = \sum_k k^2 n_k \ , \qquad (12)$$

and the number of clusters

$$N_c = \sum_k n_k \ . \qquad (13)$$

A. Results for $s = 2$

In Fig.1 we show the Binder parameter for the case $s = 2$ for system sizes $L = 10 \div 60$. It is possible to see that V for small sizes has a minimum at $k_B T/J \simeq 3.0$ and that for grater sizes it becomes constant for all temperatures, reveling a second order phase transition. Therefore we can make standard scaling analysis [20] for the thermodynamical quantities.

In particular, by definition of critical exponents ν it is

$$\xi \sim |T - T_s|^{-\nu} \qquad (14)$$

where ξ is the correlation length and $T_s = \lim_{L \to \infty} T_s(L)$ with $T_s(L)$ finite size critical temperature of the PSG model. Analogously from the definitions of the other critical exponents β, γ and α we get

$$M \sim |T - T_s|^\beta \sim \xi^{-\beta/\nu} \ , \qquad (15)$$

$$\chi \sim |T - T_s|^{-\gamma} \sim \xi^{\gamma/\nu} \ , \qquad (16)$$

$$C_H \sim |T - T_s|^{-\alpha} \sim \xi^{\alpha/\nu} \ . \qquad (17)$$

From standard scaling analysis applied to finite systems [20] we expect for M

$$M \sim L^{-\beta/\nu} f_M((T - T_s) L^{1/\nu}) \qquad (18)$$

where $f_M(x)$ is an universal function of the dimensionless variable x. Analogous scaling functions are expected for the other thermodynamical quantities. Tuning the values of critical exponents and T_s it is possible to verify the scaling hypothesis, as eq.(18), from the MC data. The values for which the data collapse give the estimates of the critical exponents and of T_s.

In Figs. 2, 3, 4 we show the data collapses for system sizes $L = 10 \div 60$. The estimated scaling parameters are [21] given in Tab. I

The estimated critical exponents are compatible, within the errors, with the expected values for a Potts model with $s = 2$ state, $i.e.$ an Ising model, in 2D: $\alpha = 0$, $\beta = 1/8 = 0.125$, $\gamma = 7/4 = 1.75$ and $\nu = 1$ [22]. The estimated critical temperature is $k_B T_s/J = 2.95 \pm 0.15$.

For the FK-CK percolation quantities we have described the critical behavior of P and S introducing a percolation set of critical exponents (α_p, β_p, γ_p and ν_p) defined by the relations

$$\xi_p \sim |T - T_p|^{-\nu_p} \qquad (19)$$

where ξ_p is the connectedness length of the clusters and $T_p = \lim_{L \to \infty} T_p(L)$ with $T_p(L)$ finite size percolation temperature,

$$P \sim |T - T_p|^{\beta_p} \sim \xi_p^{-\beta_p/\nu_p} \ , \qquad (20)$$

$$S \sim |T - T_p|^{-\gamma_p} \sim \xi_p^{\gamma_p/\nu_p} \ . \qquad (21)$$

Standard scaling analysis for finite systems [23] is applied also in this case and the results are summarized in Figs. 5, 6. The estimated scaling parameters are given in Tab. II.

All the estimated exponents are compatible, within the errors, to the corresponding thermodynamical parameters for the 2D Ising model and the numerical estimate for the percolation temperature is $k_B T_p/J = 2.925 \pm 0.075$ consistent with the estimates of T_s.

B. Results for $s = 7$ and $s = 50$

In Fig. 7 we show the Binder parameter V for the PSG model with $s = 7$ and $s = 50$ for system sizes $L = 10 \div 50$. The fact that V has a non vanishing minimum for every size revels that there is a first order phase transition. In this case there is no diverging length, therefore the scaling analysis cannot be applied. This kind of transition is characterized by the finite size relations [20]

$$C_H(T_s(L), L) \simeq \max_T(C_H(T, L)) \sim L^D \qquad (22)$$

(where D is the Euclidean dimension) for the maximum of finite size $C_H(L)$ (see Tab. III) and the relation

$$T_{\max}(L) - T_{\max}(\infty) \sim L^{-D} \qquad (23)$$

where $T_{\max}(L)$ is the temperature of the maximum of C_H (or of the mean cluster size S) for the size L and $T_{\max}(\infty)$ is the corresponding value in the thermodynamical limit, *i.e.* the corresponding transition temperature $T_s(s)$ (or $T_p(s)$). Therefore $T_s(s)$ and $T_p(s)$ can be evaluated by linear fits with one free parameter. The data are given in Tab. IV and the results are for $s = 7$ $T_s = T_p = 7.5 \pm 0.1$ and for $s = 50$ $T_s = T_p = 35.0 \pm 0.1$.

The results are summarized in the phase diagram in Fig.8: For every s the high temperature phase is disordered and non percolating; decreasing the temperature there is a second or first order phase transition (depending on s) at T_p corresponding to the percolation of FK-CK clusters and to the ordering of Potts variables; at lower temperature there is the SG transition (that in 2D occurs at $T = 0$ [18]). For a fixed realization of $\{\epsilon_{ij}\}$ it is possible to show [24] that each critical point is characterized by a diverging critical length. At $T_p(s)$ diverges the linear size of FK-CK clusters associated with the pair connectedness function, while at $T_{SG}(s)$ diverges the linear size of correlated regions.

It is interesting to note that the behavior of $T_p(s)$ can be obtained from the exact expression of the transition temperature of a ferromagnetic $2s$-state Potts model [13] only by renormalizing the number of states, *i.e.*

$$\frac{T_p}{2sa} = \frac{1}{\ln(1 + \sqrt{2sa})} \qquad (24)$$

with $a = 0.803 \pm 0.003$ (choosing $J = k_B$).

IV. COMPARISON WITH SPIN FLIP DYNAMICS

The MC dynamics used to study the equilibrium properties of the PSG model is the Swendsen and Wang (SW) cluster MC dynamics. The SW dynamics is performed in two steps. The first steps is to construct the FK-CK cluster configuration C, given an Ising and a Potts spin configuration $\{S_i, \sigma_i\}$, activating bonds with the probability in eq. (2) between NN sites when both Ising and Potts spins satisfy the interaction and with zero probability otherwise. The second step consists in reversing all the spins in a cluster at the same time with probability $1/2$, for each cluster. The sequence of the first and of the second steps applied to the whole system constitutes a MC step, that is the chosen unit of time.

This dynamic completely overcomes the problem of critical slowing down for the unfrustrated spin models [14], while turns out to suffer of diverging correlation time if applied to frustrated systems near a critical point. This inefficiency is a consequence of the fact that the FK-CK clusters used in the SW dynamics do not represent anymore in frustrated models the regions of correlated spins near a critical point [12] and their percolation temperature T_p is greater than the critical temperature. In particular, this is true for the SG model for which an efficient cluster MC dynamics does not yet exist except for 2D [26], while efficient cluster dynamics have been proposed for systems with frustration but without disorder [27,28].

Nevertheless in SG for temperatures well above the critical temperature T_{SG} and near T_p the SW dynamics is still efficient, consistently with the general observation that the cluster dynamics are efficient at least for temperatures above the percolation temperature [28].

On the other hand in Ref. [15] we have shown that the local spin-flip (SF) MC dynamics [20] in the case of PSG model with $s = 2$ is characterized near T_p by diverging correlation times. To compare the efficiency of SW dynamics to that of SF dynamics we have studied for the case $s = 2$ the correlation functions at the equilibrium, that for a generic observables A is defined as

$$f_A(t) = \left[\frac{\langle \delta(t + t_0)\delta(t_0) \rangle}{\langle \delta(t_0)^2 \rangle} \right], \qquad (25)$$

where $\delta(t) = A(t) - \langle A \rangle$ and t_0 is the equilibration time. As observables we have choose the Potts order parameter M and the energy E of the whole system.

We have also studied the time dependent nonlinear susceptibility for a quenched interaction configuration

$$\chi_{SG}(t) = \frac{1}{N} \left\langle \left[\sum_i S_i(t + t_0)S_i(t_0) \right]^2 \right\rangle \qquad (26)$$

(where N is the total number of spins). The normalized correlation function is

$$f_\chi = \frac{\chi_{SG}(t) - \chi_{SG}(t = \infty)}{\chi_{SG}(0) - \chi_{SG}(t = \infty)} \qquad (27)$$

with $\chi_{SG}(0) = N$.

For both the SW and the SF dynamics we have measured the integral correlation time defined as

$$\tau_{int} = \lim_{t_{max} \to \infty} \frac{1}{2} + \sum_{t=0}^{t_{max}} f(t) \qquad (28)$$

where f is the generic correlation function. We have considered systems with lattice sizes $L \le 30$ at temperatures above and below T_p. The data for the SF dynamics are averaged over 32 different quenched interaction configurations, since the local updating of this dynamics strongly depends on the local fluctuation of the frustration. On the other hand the results on global SW dynamics turns out to be "robust" respect the interaction configuration average, in the sense that the fluctuations of τ_{int} are within the errors estimated on the basis of a single interaction configuration analysis.

The simulations have been done with an annealing method, *i.e.* with a slow cooling of the system at each temperature. For the SW cluster dynamics $5 \cdot 10^3$ MC steps turn out to be enough to equilibrate the system at the considered temperatures and the averages are done using the the data for the following $5 \cdot 10^4$ MC steps. For the SF dynamics we have discarded the first 10^4 MC steps (defined as the local update of any spin in the system) and recorded the data for $5 \cdot 10^5$ MC steps.

In Tab. V we show the results for $L = 30$. Analogous results have been found for smaller systems.

The data show that, while the SF correlation times for M and E grow abruptly near T_p where both thermodynamical and percolation transition occur, the SW correlation times only show a slow trend to increase for decreasing temperatures, being smaller than the correspondent SF data at least of an order of magnitude. Even for τ_χ, that for SF starts to be non-zero below T_p, the SW dynamics shows smaller correlation times. For temperatures well below T_p it is possible to see that the SW dynamics is characterized by long autocorrelation times, as the SF dynamics. Therefore, at least for temperatures not too much below T_p, the SW dynamics turns out to be more efficient than local SF dynamics. In particular, near the thermodynamical transition at T_p the SW dynamics completely overcomes the critical slowing down problem in the PSG model, as for the unfrustrated models, even if frustration is present via the random interactions of the Ising spins.

V. CONCLUSIONS

Precursor phenomena characterize the paramagnetic phase of spin glass. In particular, experiments and local Monte Carlo simulations show the presence of stretched exponential autocorrelation functions well above the SG transition temperature T_{SG} [7]. The relation of the onset T^* of these precursor phenomena to any thermodynamical transition and the localization of T^* are still matters of debate [7,15]. Many numerical evidences on disordered [7] and deterministic frustrated models [10,11] have shown that T^* is consistent with the percolation temperature T_p of the the Kasteleyn - Fortuin and Coniglio - Klein clusters. In particular, a generalization of the $\pm J$ Ising SG to a $2s$-state Potts SG, that recovers the SG for $s = 1$, has shown that for $s = 2$, as for $s = 1$ [7], the $T^* = T_p$ hypothesis is numerically verified [15]. In this paper, using very efficient cluster Monte Carlo dynamics, we have shown that in 2D for any $s \neq 1$ the percolation transition corresponds to a real thermodynamical transition in the universality class of the s-state ferromagnetic Potts model. In particular, we have considered the case $s = 2$ and the cases $s = 7$, 50 where at $T_p(s)$ a second order and, respectively, a first order phase transition occurs. Exact renormalization group calculations on hierarchical lattices [16] and mean field analysis for a version

of the model without disorder [17] have shown the same scenario.

All these results suggest that the percolation transition may play a role in the precursor phenomena even in the SG case ($s = 1$), where no thermodynamical transition occurs at T_p. This idea arises from the observation that, even in SG, below T_p the frustration starts to be manifested on all the length scales by means of the FK-CK clusters, that cannot include frustrated loops. Therefore the scenario presented is that for $s \neq 1$ at the percolation temperature T_p there is a thermodynamical transition with associated dynamical anomalies that "vanishes" for $s = 1$ leaving the dynamical behavior unchanged.

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TABLE V. Integral correlation times for a PSG model with $s = 2$ and $L = 30$ for Swendsen and Wang (SW) cluster MC dynamics and local spin-flip (SF) MC dynamics for temperatures above and below $T_p(L = 30) \simeq 2.95 J/k_B$.

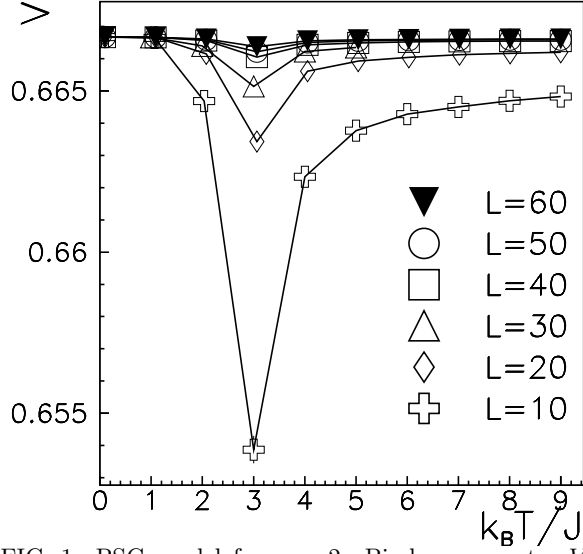| $k_B T/J$	2.75	3.00	3.25
$\tau_M(SW)$	3.08 ± 0.02	3.65 ± 0.03	1.81 ± 0.01
$\tau_M(SF)$	76.8 ± 0.3	575.8 ± 0.6	180.3 ± 0.2
$\tau_E(SW)$	9.32 ± 0.02	8.53 ± 0.07	4.45 ± 0.04
$\tau_E(SF)$	19.0 ± 0.1	67.5 ± 0.1	20.9 ± 0.5
$\tau_\chi(SW)$	9.93 ± 0.07	2.423 ± 0.002	1.585 ± 0.001
$\tau_\chi(SF)$	16.89 ± 0.02	7.06 ± 0.03	3.97 ± 0.01

TABLE I. Estimated critical exponents and critical temperatures T_s for thermodynamical quantities M, χ and C_H for the PSG model with $s = 2$.

	ν	α/ν	β/ν	γ/ν	$k_B T_s/J$
M	0.9 ± 0.3		0.13 ± 0.06		3.0 ± 0.1
χ	1.0 ± 0.3			1.75 ± 0.10	2.9 ± 0.1
C_H	1.0 ± 0.5	0.0 ± 0.1			2.9 ± 0.1

TABLE II. Estimated percolation exponents and percolation temperature T_p for P and S for the PSG model with $s = 2$.

	ν_p	β_p/ν_p	γ_p/ν_p	$k_B T_p/J$
P	0.9 ± 0.2	0.10 ± 0.03		2.95 ± 0.05
S	0.95 ± 0.15		1.6 ± 0.2	2.90 ± 0.05

TABLE III. Maxima of C_H for $s = 7$ and $s = 50$ for $L = 30, 40, 50$.

L	30	40	50
$\max_T C_H(s = 7)/L^2$	27 ± 3	26 ± 3	33 ± 3
$\max_T C_H(s = 50)/L^2$	7.4 ± 0.6	8.4 ± 0.9	7.3 ± 0.7

TABLE IV. Temperatures (in J/k_B units) of maxima of C_H and S for $s = 7$ and $s = 50$ for $L = 10, 20, 30, 40, 50$.

L	10	20	30	40	50
$T_{\max \, C_H}(s = 7)$	7.8	7.6	7.5	7.5	7.5
$T_{\max \, S}(s = 7)$	7.9	7.7	7.5	7.5	7.52
$T_{\max \, C_H}(s = 50)$	36.56	35.57	34.92	34.90	35.00
$T_{\max \, S}(s = 50)$	36.56	35.60	35.00	35.00	35.00

FIG. 1. PSG model for $s = 2$: Binder parameter V vs. dimensionless temperature $k_B T/J$ for $L = 10 \div 60$. Errors are smaller than the symbol sizes. Lines are only guides for the eyes.
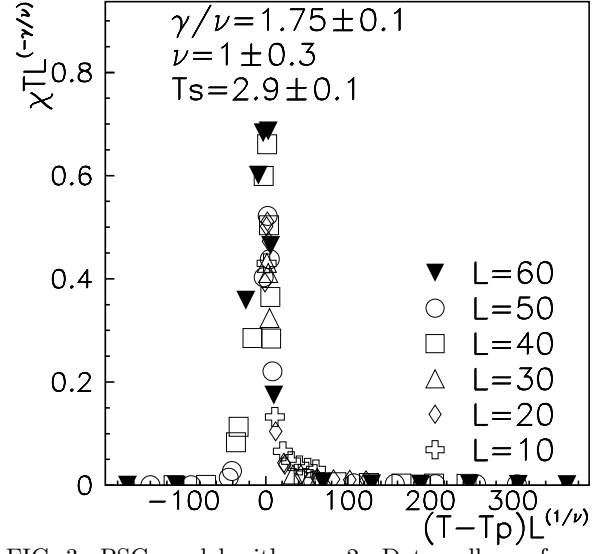

FIG. 3. PSG model with $s = 2$: Data collapse for χ for systems sizes $L = 10 \div 60$. Temperatures ate in J/k_B units. The scaling parameters are given in the figure.
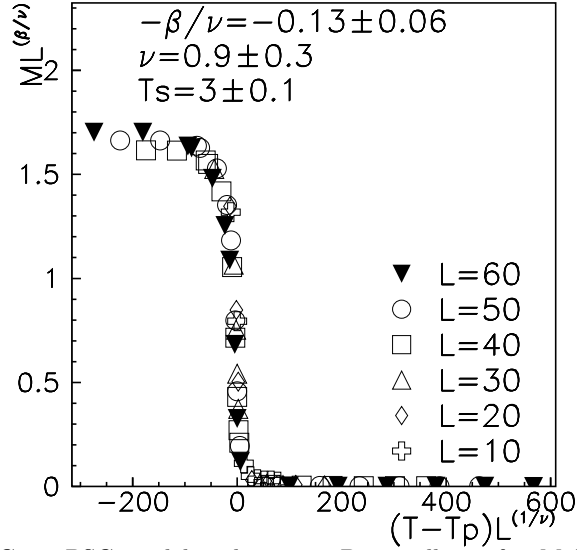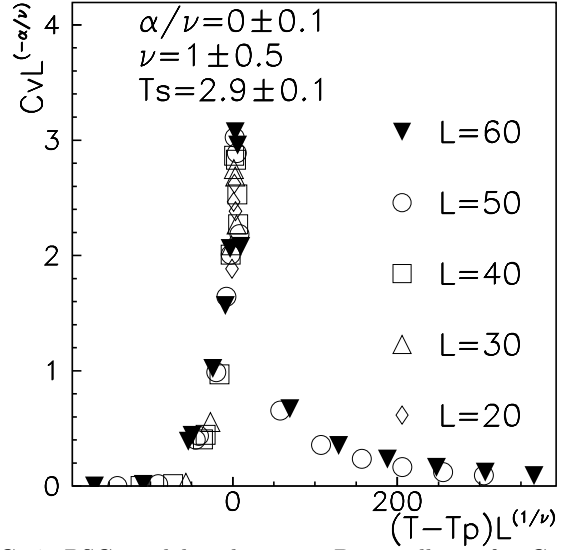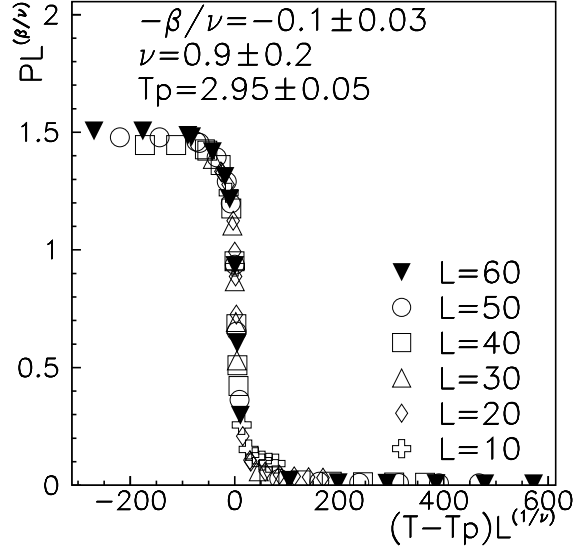

FIG. 2. PSG model with $s = 2$: Data collapse for M for systems sizes $L = 10 \div 60$. Temperatures ate in J/k_B units. The scaling parameters are given in the figure.

FIG. 4. PSG model with $s = 2$: Data collapse for C_H for systems sizes $L = 10 \div 60$. Temperatures ate in J/k_B units. The scaling parameters are given in the figure.

FIG. 5.   PSG model with $s = 2$: Data collapses for P for systems sizes $L = 10 \div 60$. Temperatures are in J/k_B units. The scaling parameters are given in the figure. In figure indexes p are omitted for the critical exponents.
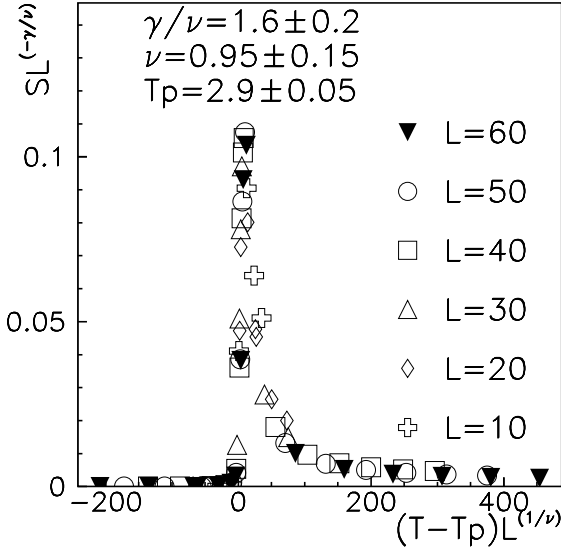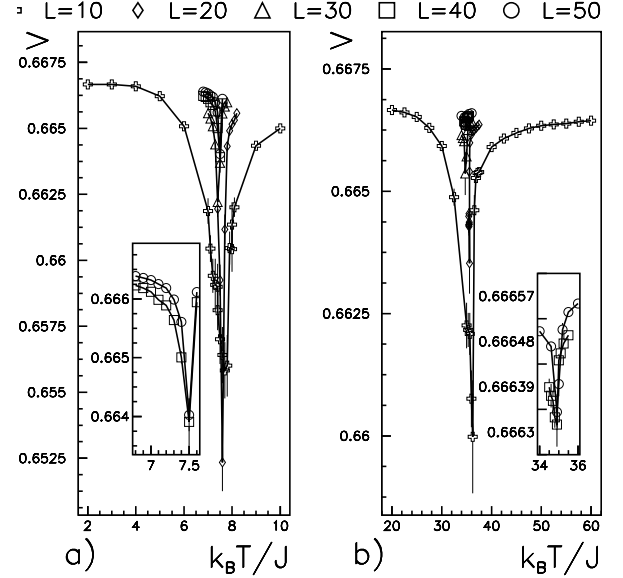

FIG. 7.   PSG model: Binder parameter V vs. dimensionless temperature $k_B T/J$ for (a) $s = 7$ and (b) $s = 50$, for $L = 10 \div 50$. Insets show the particulars for $L = 40$, 50. Where not shown, the errors are smaller than the symbol sizes. Lines are only guides for the eyes.

FIG. 6.   PSG model with $s = 2$: Data collapses for S for systems sizes $L = 10 \div 60$. Temperatures are in J/k_B units. The scaling parameters are given in the figure. In figure indexes p are omitted for the critical exponents.
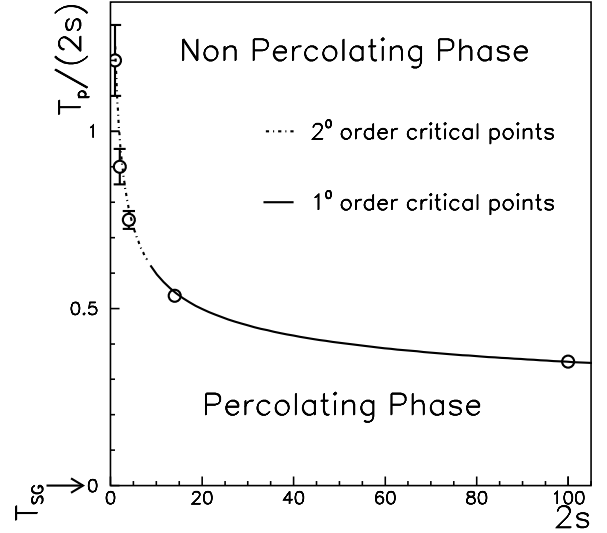

FIG. 8. PSG model: Numerical phase diagram in 2D. The data are fitted with $T_p/(2s) = a/(\ln(1 + \sqrt{2sa}))$ with $a = 0.803 \pm 0.003$ (choosing $J = k_B$). Data for $s = 1/2$ and $s = 1$ are from Ref.[25]. Where not shown, the errors are smaller than the symbol sizes.